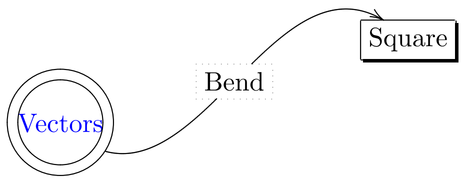

1 Homogeneous Transformations

The study of robot manipulation is concerned with the relationship between objects, and between objects and manipulators. In this chapter we will develop the representation necessary to describe these relationships. Similar problems of representation have already been solved in the field of computer graphics, where the relationship between objects must also be described. Homogeneous transformations are used in this field and in computer vision [Duda] [Robserts63] [Roberts65]. These transformations were employed by Denavit to describe linkages [Denavit] and are now used to describe manipulators [Pieper] [Paul72] [Paul77b].

We will first establish notation for vectors and planes and then introduce transformations on them. These transformations consist primarily of translation and rotation. We will then show that these transformations can also be considered as coordinate frames in which to represent objects, including the manipulator. The inverse transformation will then be introduced. A later section describes the general rotation transformation representing a rotation about a vector. An algorithm is then described to find the equivalent axis and angle of rotations represented by any given transformation. A brief section on stretching and scaling transforms is included together with a section on the perspective transformation. The chapter concludes with a section on transformation equations.

2 Notation

In describing the relationship between objects we will make use of point vectors, planes, and coordinate frames. Point vectors are denoted by lower case, bold face characters. Planes are denoted by script characters, and coordinate frames by upper case, bold face characters. For example:

vectors	$\mathbf{v}$, $\mathbf{x1}$, $\mathbf{x}$
planes	$\mathcal{P}$, $\mathcal{Q}$
coordinate frames	$\mathbf{I}$, $\mathbf{A}$, $\mathbf{CONV}$

We will use point vectors, planes, and coordinate frames as variables which have associated values. For example, a point vector has as value its three Cartesian coordinate components.

If we wish to describe a point in space, which we will call p, with respect to a coordinate frame $\mathbf{E}$, we will use a vector which we will call $\mathbf{v}$. We will write this as

$$^{E}\mathbf{v}$$

The leading superscript describes the defining coordinate frame.

We might also wish to describe this same point, p, with respect to a different coordinate frame, for example $\mathbf{H}$, using a vector $\mathbf{w}$ as

$$^{H}\mathbf{w}$$

$\mathbf{v}$ and $\mathbf{w}$ are two vectors which probably have different component values and $\mathbf{v} \neq \mathbf{w}$ even though both vectors describe the same point p. The case might also exist of a vector $\mathbf{a}$ describing a point 3 inches above any frame

$$^{F^1}\mathbf{a} \qquad ^{F^2}\mathbf{a}$$

In this case the vectors are identical but describe different points. Frequently, the defining frame will be obvious from the text and the superscripts will be left off. In many cases the name of the vector will be the same as the name of the object described, for example, the tip of a pin might be described by a vector **tip** with respect to a frame **BASE** as

$$^{BASE}\mathbf{tip}$$

If it were obvious from the text that we were describing the vector with respect to **BASE** then we might simply write

$$\mathbf{tip}$$

If we also wish to describe this point with respect to another coordinate frame say, **HAND**, then we must use another vector to describe this relationship, for example

$$^{HAND}\mathbf{tv}$$

$^{HAND}\mathbf{tv}$ and **tip** both describe the same feature but have different values. In order to refer to individual components of coordinate frames, point vectors, or planes, we add subscripts to indicate the particular component. For example, the vector $^{HAND}\mathbf{tv}$ has components $^{HAND}\mathbf{tv}_\mathbf{x}$, $^{HAND}\mathbf{tv}_\mathbf{y}$, $^{HAND}\mathbf{tv}_\mathbf{z}$.

3 Vectors

The homogeneous coordinate representation of objects in n-space is an $(n+1)$-space entity such that a particular perspective projection recreates the n-space. This can also be viewed as the addition of an extra coordinate to each vector, a scale factor, such that the vector has the same meaning if each component, including the scale factor, is multiplied by a constant.

A point vector

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \tag{1.1}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the x, y, and z coordinate axes, respectively, is represented in homogeneous coordinates as a column matrix

$$\mathbf{v} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \\ \mathbf{w} \end{bmatrix} \tag{1.2}$$

where

$$\begin{aligned} \mathbf{a} &= \mathbf{x}/\mathbf{w} \\ \mathbf{b} &= \mathbf{y}/\mathbf{w} \\ \mathbf{c} &= \mathbf{z}/\mathbf{w} \end{aligned} \tag{1.3}$$

Thus the vector $3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ can be represented as $[3, 4, 5, 1]^{\mathrm{T}}$ or as $[6, 8, 10, 2]^{\mathrm{T}}$ or again as $[-30, -40, -50, -10]^{\mathrm{T}}$, etc. The superscript T indicates the transpose of the row vector into a column vector. The vector at the origin, the null vector, is represented as $[0, 0, 0, n]^{\mathrm{T}}$ where n is any non-zero scale factor. The vector $[0, 0, 0, 0]^{\mathrm{T}}$ is undefined. Vectors of the form $[a, b, c, 0]^{\mathrm{T}}$ represent vectors at infinity and are used to represent directions; the addition of any other finite vector does not change their value in any way.

We will also make use of the vector dot and cross products. Given two vectors

$$\begin{aligned} \mathbf{a} &= a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \\ \mathbf{b} &= b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k} \end{aligned} \tag{1.4}$$

we define the vector dot product, indicated by "·" as

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \tag{1.5}$$

The dot product of two vectors is a scalar. The cross product, indicated by an "×", is another vector perpendicular to the plane formed by the vectors of the product and is defined by

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} \tag{1.6}$$

This definition is easily remembered as the expansion of the determinant

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1.7}$$

4 Planes

A plane is represented as a row matrix

$$\mathcal{P} = [a, b, c, d] \tag{1.8}$$

such that if a point $\mathbf{v}$ lies in a plane $\mathcal{P}$ the matrix product

$$\mathcal{P}\ \mathbf{v} = 0 \tag{1.9}$$

or in expanded form

$$xa + yb + zc + wd = 0 \tag{1.10}$$

If we define a constant

$$m = +\sqrt{a^2 + b^2 + c^2} \tag{1.11}$$

and divide Equation 1.10 by wm we obtain

$$\frac{x}{w}\frac{a}{m} + \frac{y}{w}\frac{b}{m} + \frac{z}{w}\frac{c}{m} = -\frac{d}{m} \tag{1.12}$$

The left hand side of Equation 1.12 is the vector dot product of two vectors $(x/w)\mathbf{i} + (y/w)\mathbf{j} + (z/w)\mathbf{k}$ and $(a/m)\mathbf{i} + (b/m)\mathbf{j} + (c/m)\mathbf{k}$ and represents the directed distance of the point $(x/w)\mathbf{i} + (y/w)\mathbf{j} + (z/w)\mathbf{k}$ along the vector $(a/m)\mathbf{i} + (b/m)\mathbf{j} + (c/m)\mathbf{k}$. The vector $(a/m)\mathbf{i} + (b/m)\mathbf{j} + (c/m)\mathbf{k}$ can be interpreted as the outward pointing normal of a plane situated a distance $-d/m$ from the origin in the direction of the normal. Thus a plane $\mathcal{P}$ parallel to the x,y plane, one unit along the z axis, is represented as

$$\mathcal{P} = [0, 0, 1, -1] \tag{1.13}$$

$$\text{or as} \quad \mathcal{P} = [0, 0, 2, -2] \tag{1.14}$$

$$\text{or as} \quad \mathcal{P} = [0, 0, -100, 100] \tag{1.15}$$

A point $\mathbf{v} = [10, 20, 1, 1]$ should lie in this plane

$$[0, 0, -100, 100] \begin{bmatrix} 10 \\ 20 \\ 1 \\ 1 \end{bmatrix} = 0 \tag{1.16}$$

or

$$[0, 0, 1, -1] \begin{bmatrix} -5 \\ -10 \\ -.5 \\ -.5 \end{bmatrix} = 0 \tag{1.17}$$

The point $\mathbf{v} = [0, 0, 2, 1]$ lies above the plane

$$[0, 0, 2, -2] \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} = 2 \tag{1.18}$$

and $\mathcal{P}\ \mathbf{v}$ is indeed positive, indicating that the point is outside the plane in the direction of the outward pointing normal. A point $\mathbf{v} = [0, 0, 0, 1]$ lies below the plane

$$[0, 0, 1, -1] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = -1 \tag{1.19}$$

The plane $[0, 0, 0, 0]$ is undefined.