Outline of Hydrodynamics Algorithm

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1 Equations

1.1 Momentum Equations

integral and corner force form

$$M_p \frac{d\vec{v}_p}{dt} = -\int_{V_p} \vec{\nabla} P dV = -\oint_b P d\vec{S} = \sum_z \vec{f}_z^p \equiv \vec{F}_p \tag{7}$$

discrete form

$$M_p \Delta \vec{v}_p = \sum_z \vec{f}_z^{p,\sigma} \Delta t \tag{17}$$

1.2 Specific Internal Energy Equations

corner force form

$$M_z \frac{de_z}{dt} = -\sum_p \vec{f}_p^z \cdot \vec{v}_p \tag{12}$$

coordinate form

$$M_z \frac{de_z}{dt} = -P_z \left((\vec{a}_8 + \vec{a}_1) \cdot \vec{v}_1 + (\vec{a}_2 + \vec{a}_3) \cdot \vec{v}_2 + (\vec{a}_4 + \vec{a}_5) \cdot \vec{v}_3 + (\vec{a}_6 + \vec{a}_7) \cdot \vec{v}_4 \right) \tag{13}$$

discrete form

$$\Delta e_z = -\sum_p \vec{f}_p^{z,\sigma} \cdot \vec{v}_p^{n+1/2} \Delta t / M_z \tag{20}$$

1.3 Energy Equations

time-dependent form

$$E_T(t) = E_T(0) + \sum_{m=1\ldots n} \Delta t_m \sum_i \vec{f}_{bd,i}^{\sigma} \cdot \vec{v}_{bd,i}^{m+1/2} \tag{15}$$

2 Algorithm

2.1 Don Marshall's Memo

- Use xy-Cartesian geometry.
- The mesh will have five radial lines in $\pi/2$ radians, i.e., four cells in the angular direction. Use ten equally spaced radial shells with an outermost radius of 1.0.
- For the pressure P, use gamma law (ideal gas) equation of state

  $$P = (\gamma - 1)\rho e.$$

  γ is a constant, e.g., 5/3 or 4/3.
- initial $e_z = 1$
- initial $\rho_z = 1$
- initial $\vec{v}_p = \vec{0}$
- Use a predictor corrector method.

2.2 Mesh Values

value	location
corner force $\vec{f}_p^z$	point
corner mass m_p^z	point
coordinate positions $\vec{r}_p$	point
density ρ_z	zone
internal energy e_z	zone
mass M_z	zone
pressure P_z	zone
velocity $\vec{v}_p$	point

Initially, $m_p^z = \rho_z *$ corner volume.

2.3 Constants

- γ, e.g., 4/3 or 5/3
- mass $M_z = \sum_p m_p^z$, where $m_p^z = \rho_z *$ corner volume.
- timestep $\Delta t = 0.01$, divided by two for predictor and corrector phases

2.4 Boundary Conditions

Use a velocity of $\vec{0}$ for all cell vertices at the origin. Velocity vectors along axes should have zero normal components. (I should use a Pooma ¿enforce? condition. John Hall knows more about this.)

2.5 What An Iteration Impacts

The algorithm consists of a series of iterations, one for each time period. We number each time period, e.g., n. Each iteration takes a set of values as input and performs some computations, changing some values.

values	predictor	corrector	output
coordinate positions $\vec{r}_p$	n	$n+1/2$	y
density ρ_z	n	$n+1$	y
internal energy e_z	n	$n+1$	y
mass M_z	constant	constant	n
constant timestep Δt	0.005	0.005	n
velocity $\vec{v}_p$	n	$n+1$	y

Both the predictor and corrector iterations have the same form, but they use input values from different iterations. These are listed in the above table. The "output" column has a "y" if an iteration changes the corresponding value.

The Marshalls say to use only the predictor iterations, not the corrector iterations.

2.6 Preparation Before Beginning the Iterations

Before starting the iterations, initial values for the input values must be computed.

1. Compute the coordinate positions $\vec{r}_p$. Use the mesh description listed in Section 2.1.

2. Use an initial internal energy e_z of one.

3. Use an initial pressure density ρ_z of one.

4. The initial masses M_z can be computed using the density ρ_z and the initial volume.

$$M_z = \rho_z V_z \tag{1}$$

$$V_z = \frac{1}{2}\left((r_2 - r_4)(z_3 - z_1) + (r_1 - r_3)(z_2 - z_4)\right) \tag{56}$$

5. The timestep Δt is a constant.

6. Use an arbitrary initial velocity $\vec{v}_p$, e.g., zero.

2.7 An Iteration

The steps of an iteration include:

1. Compute zone pressures P_z.

$$P_z = (\gamma - 1)\rho_z e_z$$

2. Compute the outward normals.

$$\vec{a}_{2i-1} = \vec{a}_{2i} = \frac{1}{2}(z_{i+1} - z_i)\hat{r} - \frac{1}{2}(r_{i+1} - r_i)\hat{z} \tag{60}$$

where the coordinate subscripts $i = 1, \ldots, 4$ are computed modulo 4.

3. Compute the corner forces.

$$\begin{aligned} \vec{f}_z^1 &= P_z(\vec{a}_8 + \vec{a}_1) \\ \vec{f}_z^2 &= P_z(\vec{a}_2 + \vec{a}_3) \\ \vec{f}_z^3 &= P_z(\vec{a}_4 + \vec{a}_5) \\ \vec{f}_z^4 &= P_z(\vec{a}_6 + \vec{a}_7), \end{aligned}$$

using the notation of Fig. 1.

4. Change velocity using Eq. (17).

$$\Delta\vec{v}_p = \frac{\Delta t}{M_p} \sum_z \vec{f}_z^{p,\sigma} \tag{17$'$}$$

Thus, one has $\vec{v}_p^n$ and $\vec{v}_p^{n+1}$.

5. Change the specific internal energy using Eq. (20).

$$\Delta e_z = -\frac{\Delta t}{M_z} \sum_p \vec{f}_p^{z,\sigma} \cdot \vec{v}_p^{n+1/2} \tag{20$'$}$$

$$\vec{v}_p^{n+1/2} = (\vec{v}_p^n + \vec{v}_p^{n+1})/2$$

6. Change coordinates.

$$\vec{r}_p \leftarrow \vec{r}_p + \vec{v}_p^{n+1/2}\Delta t$$

7. Update volume.

$$V_z = \frac{1}{2}\left((r_2 - r_4)(z_3 - z_1) + (r_1 - r_3)(z_2 - z_4)\right) \tag{56}$$

8. Update density.

$$\rho_z(t) = M_z/V_z(t)$$

2.8 Termination

For the kernel, we will just run the algorithm for a fixed number of iterations, e.g., 1 or 2 or 3.