1. Quadrics

The space you are considering is $\mathbb{P}^3$. This is what we call a manifold. A manifold is some space that locally looks like $\mathbb{R}^n$, for some n. Manifolds thus have patches, on which they have coordinates that are the local smooth bijections to $\mathbb{R}^n$. So the surface of the earth is a 2-dimensional manifold. The patches are the northern hemisphere, and the southern hemisphere.

Life is similar in $\mathbb{P}^3$, it locally looks like $\mathbb{R}^3$. A point in $\mathbb{P}^3$ is a line through the origin in $\mathbb{R}^4$. Such a line is determined by a vector, excluding $(0,0,0,0)$. In fact that vector is a little too much information because multiples λv of some vector v result in the same line. That leads to the following notation for a point in $\mathbb{P}^3$:

$$[x_0; x_1; x_2; x_3] \quad x_i \neq 0 i = 1, \cdots 4$$

The brackets notation with the semicolon is there to indicate that $[1; 2; -3; \pi]$ and $[2; 4; -6; 2\pi]$ are the same thing.

For the surface of the earth we had two charts. There are four natural charts on $\mathbb{P}^3$. Take the lines for which $x_i \neq 0$, they make up the chart U_i. For U_0 the smooth bijection is as follows:

$$[x_0; x_1; x_2; x_3] \to (\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}) \tag{1}$$

Now if my ellipsoid before had the equation

$$a_{2000} + a_{1100}x_1 + a_{1010}x_2 + a_{1001}x_3 + a_{0200}x_1^2 + a_{0020}x_2^2 + a_{0002}x_3^2 + \tag{2}$$
$$a_{0110}x_1x_2 + a_{0011}x_2x_3 + a_{0101}x_1x_3 \tag{3}$$

then it is the image under the map of equation (1) of the following points in $\mathbb{P}^3$:

$$a_{2000}x_0^2 + a_{1100}x_1x_0 + a_{1010}x_0x_2 + a_{1001}x_0x_3 + a_{0200}x_1^2 + a_{0020}x_2^2 + a_{0002}x_3^2 + \tag{4}$$
$$a_{0110}x_1x_2 + a_{0011}x_2x_3 + a_{0101}x_1x_3 \tag{5}$$

If I do a projective transformation on $\mathbb{R}^3$ as your program does then that corresponds to doing a transformation on $\mathbb{P}^4$. Obviously only linear transformations make sense here. They are given by 4x4 matrices:

$$\mathbf{B} = \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Looking closer at equation (4) we see that it can be written in matrix form as well:

$$\langle \begin{pmatrix} a_{2000} & \frac{1}{2}a_{1100} & \frac{1}{2}a_{1010} & \frac{1}{2}a_{1001} \\ \frac{1}{2}a_{1100} & a_{0200} & \frac{1}{2}a_{0110} & \frac{1}{2}a_{0101} \\ \frac{1}{2}a_{1010} & \frac{1}{2}a_{0110} & a_{0020} & \frac{1}{2}a_{0011} \\ \frac{1}{2}a_{1001} & \frac{1}{2}a_{0101} & \frac{1}{2}a_{0011} & a_{0002} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \rangle = 0 \tag{6}$$

Now if

$$[x_0'; x_1'; x_2'; x_3'] = \mathbf{B}([x_0; x_1; x_2; x_3])$$

then the image of the lines that satisfy (6) is

$$\langle (\mathbf{B}^{-1})^T \begin{pmatrix} a_{2000} & \frac{1}{2}a_{1100} & \frac{1}{2}a_{1010} & \frac{1}{2}a_{1001} \\ \frac{1}{2}a_{1100} & a_{0200} & \frac{1}{2}a_{0110} & \frac{1}{2}a_{0101} \\ \frac{1}{2}a_{1010} & \frac{1}{2}a_{0110} & a_{0020} & \frac{1}{2}a_{0011} \\ \frac{1}{2}a_{1001} & \frac{1}{2}a_{0101} & \frac{1}{2}a_{0011} & a_{0002} \end{pmatrix} \mathbf{B}^{-1}[x_0'; x_1'; x_2'; x_3'], [x_0'; x_1'; x_2'; x_3'] \rangle$$

What I am saying is that storing those ten coefficients is not such a bad idea. It is easy to compute how they change under a projective transformation.

You ask how to get the principal axes of the ellipsoid from these coefficients, and whether a projective transformation can transform an ellipsoid into something that is not an ellipsoid.

For that we move back to the original presentation of the ellipsoid. We are going to apply some linear algebra to solve the problem.

Proposition 1. *If* $\mathbf{A}$ *is a symmetric matrix if and only if, its eigenvectors, the solutions to* $\mathbf{A}\vec{v} = \lambda\vec{v}$*, are all orthogonal.*

Proof. The proof uses complex number and I'll spare you that one. □

Look at equation (2) again. It can be written in the form:

$$\langle \mathbf{A} \begin{pmatrix} x_1 - c_1 \\ x_2 - c_2 \\ x_3 - c_3 \end{pmatrix}, \begin{pmatrix} x_1 - c_1 \\ x_2 - c_2 \\ x_3 - c_3 \end{pmatrix} \rangle = 1 \tag{7}$$

The eigenvectors of $\mathbf{A}$ are the principal vectors of your ellipsoid. The point (c_1, c_2, c_3) is the center of the ellipsoid. When all the eigenvalues λ_i are positive this is indeed an ellipsoid. The square root of the inverse of the eigenvalues are the principal radii.

2. The Platonic solids

Plato subscribed to the theory that we humans are locked in a cave. The sunlight is behind us and all we can see are the shades of the real things, the ideas. These ideas are thus the perfect forms, from which the errant shades are derived. The platonic solids are such perfect forms. They show up in a lot of places, if you look a little closer. See the book "Regular Polytopes", by H.S.M. Coxeter for more information.

The platonic solids are the cube, the tetrahedron, the octahedron, the dodecahedron and the icosahedron. The platonic solids are all polytopes: convex subsets defined by linear inequalities. They are made up of vertices, edges and faces. Some mathematicians like to speak about these as 0-facets, 1-facets and 2-facets: a vertex is 0-dimensional, an edge is 1-dimensional and a face is 2-dimensional.

The perfectness of the platonic solids lies among many other things in the following properties:

- The faces are all the same regular polygons.
- All edges have the same length.
- All dihedral angles between faces are the same. (The dihedral angle that two planes in $\mathbb{R}^3$ make is the minimal angle that two normals make.)

Each of these three statements is equivalent, and each of them singles out the platonic solids as the unique polytopes having that property.

Coordinates for the cube are obviously

$$(\pm 1, \pm 1, \pm 1)$$

The tetrahedron has four faces. Its coordinates are:

$$(0, 1, 0),\ (0, 0, \sqrt{2}),\ (\frac{1}{2}\sqrt{3}, -\frac{1}{2}),\ (-\frac{1}{2}\sqrt{3}, -frac12, 0)$$

The octahedron has eight faces, and six vertices. One face is

$$\{(0,0,1),(0,1,0),(0,0,1)\}$$

The other faces can be obtained by mirroring through the three coordinate planes $x_i = 0$, $i = 1,2,3$.

For the last two it becomes handy to introduce the Euler-Poincare formula. According to this formula the number of vertices, minus the number of edges, plus the number of faces equals 2.

The dodecahedron is slightly more complicated. It has 12 faces. Those faces each have 5 edges. So five times the number of faces equals two times the number of edges. There are thus 30 edges, and by the Euler-Poincaré formula there are 20 vertices.

The faces of the dodecahedron must be regular polygons. They have 5 edges so they are regular pentagons. A regular pentagon in the plane, whose vertices lie on a circle with radius 1 centered at the origin, has vertices,

$$\left(\cos\frac{2k\pi}{5}, \sin\frac{2k\pi}{5}\right) \quad k = 0, \cdots, 4$$

We leave it to the connoisseurs to take a pentagon and check the following formula:

$$\cos\left(\frac{\pi}{5}\right) = \frac{1}{4}(\sqrt{5}+1) \tag{8}$$

Note that we use radians here. A right angle no longer is 90^o degrees, now it's $\frac{\pi}{2}$.

We use the trigonometric formulas

$$\cos^2\alpha + \sin^2\alpha = 1 \quad \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \quad \sin(2\alpha) = 2\sin\alpha\cos\alpha$$

to find that

$$\sin\left(\frac{\pi}{5}\right) = \frac{1}{4}\sqrt{10-2\sqrt{5}}\,\sin\left(\frac{2\pi}{5}\right) = \frac{1}{4}\sqrt{10-2\sqrt{5}}\,\cos\left(\frac{2\pi}{5}\right) = \frac{1}{4}(1+\sqrt{5}) \tag{9}$$

Thus we have constructed the points of a regular pentagon. We place the pentagon at height c. In that way we have ten vertices of the dodecahedron

$$\left(\cos\frac{2k\pi}{5}, \sin\frac{2k\pi}{5}, \pm c\right) \quad k = 0, \cdots, 4$$

Now we need to find the right height c.

The last platonic solid we want to study is the one with 20 faces: the icosahedron. Each of those faces has three edges. So three times the number of faces is two times the number of edges. So there are thirty edges. And according to Euler-Poincaré there are 12 vertices. The easiest way to get these is to take the vertices:

$$(0, \pm c, \pm 1) \quad (\pm 1, 0, \pm c) \quad (\pm c, \pm 1, 0)$$

Here c is again the golden ratio $\frac{1}{2}\left(1+\sqrt{5}\right)$.

3. Intersections of ellipses

Suppose we have an ellipse e and an ellipse f in the plane $\mathbb{R}^2$. We want to know where they intersect and how. Basically there are six possibilities, pictured in 1, all of which we will have to deal with.

To simplify the problem we put the ellipse E in standard form, it will just be the standard ellipse with radii a_e and b_e.

$$\frac{x_1^2}{a_e^2} + \frac{x_2^2}{b_e^2} = 1 \tag{10}$$

For the other ellipse we have two (orthogonal) principal axes v_a and v_b, with

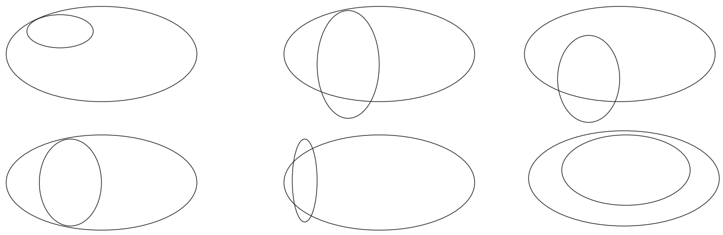

FIGURE 1. The possible intersections of two ellipses

corresponding radii a_f and b_f, and center c. We have seen in the above that eigenvectors are only determined up to a multiple: if v is an eigenvector of the matrix $\mathbf{A}$, then $2v$ is also an eigenvector. It is thus harmless to assume that $\|v_a\| = a_f^2$ and similarly $\|v_b\| = b_f^2$.

Note also that the figure 1 is a bit deceiving. In it the principal axes of e and f are parallel.

Here's a bit of terminology. The parabola is a curve in the plane. When we parameterize it, we write $\gamma(t) = (t, t^2)$. We say that γ is a *parameterization* of the parabola. When we write $x_2 - x_1^2 = 0$ we specify an *implicit equation* for it. Suppose we have a parameterization $\gamma = (\gamma_1, \gamma_2)$ of some curve, and we have another curve specified by an implicit equation $F(x_1, x_2) = 0$, then we can find their intersection points by solving $F(\gamma_1(t), \gamma_2(t)) = 0$.

The parameterization of the ellipse f is

$$t \mapsto (c + v_a \cos(t) + v_b \sin(t))$$

We now use the implicit equation (10) for the ellipse e to study the intersection.

From the equation for f we have

$$x_1 = c_1 + v_{a,1} \cos(t) + v_{b,1} \sin(t) \quad x_2 = c_2 + v_{a,2} \cos(t) + v_{b,2} \sin(t)$$

Hence

$$\begin{gathered} x_1^2 = c_1^2 + v_{a,1}^2 \cos^2(t) + v_{b,1} \sin^2(t) + \\ 2c_1 v_{a,1} \cos(t) + 2c_1 v_{b,1} \sin(t) + 2v_{a,1} v_{b,1} \cos(t) \sin(t) \end{gathered} \tag{11}$$

and

$$\begin{gathered} x_2^2 = c_2^2 + v_{a,2}^2 \cos^2(t) + v_{b,2} \sin^2(t) + \\ 2c_2 v_{a,2} \cos(t) + 2c_2 v_{b,2} \sin(t) + 2v_{a,2} v_{b,2} \cos(t) \sin(t) \end{gathered} \tag{12}$$

We insert (11) and (12) in (10), to get an expression that involves terms in $\cos^2(t)$, $\sin^2(t)$, $\cos(t)\sin(t)$, $\sin(t)$, $\cos(t)$ and some constant. Here is the complete expression:

$$\cos^2(t)\left(\frac{v_{a,1}^2}{e_a^2}+\frac{v_{a,2}^2}{e_b^2}\right)+ \tag{13}$$

$$\sin^2(t)\left(\frac{v_{b,1}^2}{e_a^2}+\frac{v_{a,2}^2}{e_b^2}\right)+ \tag{14}$$

$$\cos(t)\sin(t)\left(\frac{2v_{a,1}v_{b,1}}{e_a^2}+\frac{2v_{a,2}v_{b,2}}{e_b^2}\right)+ \tag{15}$$

$$\cos(t))\left(\frac{2v_{a,1}v_{b,1}}{e_a^2}+\frac{2v_{a,2}v_{b,2}}{e_b^2}\right)+ \tag{16}$$

$$\sin(t)\left(\frac{2v_{a,1}v_{b,1}}{e_a^2}+\frac{2v_{a,2}v_{b,2}}{e_b^2}\right)+ \tag{17}$$

$$\frac{c_1^2}{e_a^2}+\frac{c_2}{e_b^2}-1=0 \tag{18}$$

The solutions to this equation give some values for t. We can either eliminate $\cos(t)$ from the equation, or $\sin(t)$. In both cases we get 4-th degree polynomials in either $\cos(t)$ or $\sin(t)$. These can be explicitly solved, alike quadratic polynomials. If you want the explicit - very lenghty - formulas I can give them to you. In the end you'll have extremely accurate,fast code, but it won't be much fun to program.

Note that the six pictures above correspond exactly to what might happen for the zeroes of a degree four polynomial.

- There can be no zeroes: x^4+1.
- There can be one zero: x^4
- There can be two zeroes: $1-x^4$
- There can be two “tangent” zeroes: $(x^2-1)^2$
- There can be three zeroes: $(x-1)^2(x-2)(x-3)$
- There can be four zeroes: $x(x-1)(x-2)(x-3)$

Thus that we end up with a degree four polynomial is not very surprising.

It would be advantageous to know in advance what sort of intersections there are. In particular, if the ellipses e and f become tangent somewhere we would like to know this in advance. Numerically tangent points are rather ugly beasts. That is because two curves having a pair of very close intersection points are almost tangent. In fact, would one take “random” curves, then they would be tangent with probability 0.

As there can be at most two points where the two ellipses are tangent - where there more then the ellipses are the same - we have good hopes that these equations result in much less unwieldly computations.

4. Perspective and transforms